

CHARACTERIZATION OF ENTROPY FOR SPACING SHIFTS

DAWOUD AHMADI DASTJERDI AND MALIHEH DABBAGHIAN AMIRI

ABSTRACT. Suppose $P \subseteq \mathbb{N}$ and let (Σ_P, σ_P) be the space of a spacing shift. We show that if entropy $h_{\sigma_P} = 0$ then (Σ_P, σ_P) is proximal. Also $h_{\sigma_P} = 0$ if and only if $P = \mathbb{N} \setminus E$ where E is an intersective set. Moreover, we show that $h_{\sigma_P} > 0$ implies that P is a Δ^* set; and by giving a class of examples, we show that this is not a sufficient condition. Then there is enough results to solve question 5 given in [J. Banks et al., *Dynamics of Spacing Shifts*, Discrete Contin. Dyn. Syst., to appear.].

INTRODUCTION AND DEFINITIONS

In this paper we give a characterization of entropy of a spacing shifts by the combinatorial property of the set $P \subseteq \mathbb{N}$ which defines a spacing shift. A detailed study for spacing shifts can be found in [1], so we here only consider the basic definitions and notions needed for our task.

A topological dynamical system (TDS) is a pair (X, T) such that X is a compact metric space and T is a continuous surjective self map. The *orbit closure* of a point x in (X, T) is the set $\overline{\mathcal{O}}(x) = \overline{\{T^n(x) : n \in \mathbb{N}\}}$. A system (X, T) is *transitive* if it has a point x such that $\overline{\mathcal{O}}(x) = X$. Also a point x is *recurrent* if for every neighborhood U of x there exists $n \neq 0$ such that $T^n(x) \in U$. We let $N(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ and $N(U, V) = \{n \in \mathbb{N} : T^n(U) \cap V \neq \emptyset\}$ where U and V are open sets.

Let $x_1, x_2 \in X$. One says that $(x_1, x_2) \in X \times X$ is a *proximal pair* if

$$\liminf_{n \rightarrow \infty} d(T^n(x), T^n(y)) = 0;$$

and a TDS is called *proximal* if all $(x_1, x_2) \in X \times X$ are proximal pairs.

Let $A = \{a_n\}_{n \in \mathbb{N}}$ be an increasing sequence of natural numbers. Then $s = a_{i_1} + a_{i_2} + \dots + a_{i_n}$, $i_j < i_{j+1}$ is called a *partial finite sum* of A . The *finite sums* of A denoted by $FS(A)$ is the set of all partial finite sums. A set $F \subset \mathbb{N}$ is called *IP-set* if it contains the finite sums of some sequence of natural numbers. Let \mathcal{IP} be the set of all *IP*-sets.

A set $D \subset \mathbb{N}$ is called Δ -set if there exists an increasing sequence of natural numbers $S = (s_n)_{n \in \mathbb{N}}$ such that the difference set $\Delta(S) = \{s_i - s_j : i > j\} \subset D$. Denote by Δ the set of all Δ -sets. Any *IP*-set is a Δ -set; for let $S = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots\}$.

A collection \mathcal{F} of non-empty subsets of \mathbb{N} is called a *family* if it is hereditary upward: if $F \in \mathcal{F}$ and $F \subset F'$, then $F' \in \mathcal{F}$. The dual family \mathcal{F}^* , is defined to be all subsets of \mathbb{N} that meets all sets in \mathcal{F} . That is

$$\mathcal{F}^* = \{G \subset \mathbb{N} : G \cap F \neq \emptyset, \forall F \in \mathcal{F}\}.$$

2010 *Mathematics Subject Classification.* Primary 37B10; Secondary 37B40, 37B20, 37B05.

Key words and phrases. entropy, proximal, Δ^* set, *IP*-set, density.

Hence \mathcal{IP}^* and Δ^* are the dual family of \mathcal{IP} and Δ respectively.

The notions for a subset of natural numbers such as Δ or IP are structural notions. For instance, an IP -set is more structured than a Δ -set. Other structures are also defined [7], [3]. There are also notions for largeness which are defined by means of different densities on subsets of natural numbers. See [7], [2] for a rather complete treatment for both of these notions. Let $A \subseteq \mathbb{N}$. Then

$$\overline{d}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n}$$

is called the *upper density* of A . Also the *lower density* is defined as

$$\underline{d}(A) = \liminf_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

When $\overline{d}(A) = \underline{d}(A)$ then it is called the *density* of A and is denoted by $d(A)$. The *upper Banach density* of A is denoted by $d^*(A)$ and is defined as

$$d^*(A) = \limsup_{N_i - M_i \rightarrow \infty} \frac{|A \cap \{M_i, M_i + 1, \dots, N_i\}|}{N_i - M_i + 1}.$$

When there is $k \in \mathbb{N}$ such that all the intervals in $\mathbb{N} \setminus A$ have length less than k , then A is called *syndetic*. The length of the largest of such intervals will be called the *gap* of A . Clearly, $\underline{d}(A) > 0$ for any syndetic set A . The dual of syndetic sets are *thick* sets; a set is thick if and only if $d^*(A) = 1$. We say A is *thickly syndetic* if for every N the positions where consecutive elements of length N begins form a syndetic set.

Note that Δ^* -sets are highly structured and are syndetic [3]. Another of such large and structured subsets of \mathbb{N} are *Bohr sets*. We say that a subset $A \subset \mathbb{N}$ is a Bohr set if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and open set $U \subset \mathbb{T}^m$ such that

$$\{n \in \mathbb{N} : n\alpha \in U\}$$

is in A . In particular, every $k\mathbb{N}$ is a Bohr set.

Definition 0.1. For any set $P \subset \mathbb{N}$ define a *spacing shift* to be the subshift

$$\Sigma_P = \{s \in \Sigma : s_i = s_j = 1 \Rightarrow |i - j| \in P \cup \{0\}\}.$$

For any $y \in \Sigma_P$ we associate a set $A_y = \{i : y_i = 1\}$. it is clear that $A_y - A_y \subset P$. Therefore, notions of largeness and structure for A_y gives the same notions for incidence of 1's for y . That is we set

$$d(y) := d(A_y) = \lim_{n \rightarrow \infty} \frac{\sum_1^n y_i}{n} = \lim_{n \rightarrow \infty} \frac{|A_y \cap \{1, \dots, n\}|}{n}.$$

Similarly, $\overline{d}(y)$, $\underline{d}(y)$ and $d^*(y)$ can be defined.

By Definition 0.1, it is clear that $A_y - A_y \subset P$.

Acknowledgements. We would like to thank Maryam Hosseini for her fruitful discussions.

1. ZERO ENTROPY GIVES PROXIMALITY

The following questions arises in [1, Question 5].

“Is there P such that $\mathbb{N} \setminus P$ does not contain IP -set but Σ_P is proximal? What about positive topological entropy? Are these two properties (i.e proximal and zero entropy) essentially different in the context of spacing shifts? ”

We give positive answer to the first question but we will show that if $\mathbb{N} \setminus P$ contains Δ -set (and hence IP -set), then the entropy is zero. Also we will show that zero entropy in spacing shifts implies proximality.

For any $x, y \in \Sigma_P$ let

$$F_{xy}(t) = \liminf_{n \rightarrow \infty} \frac{1}{n} |\{0 \leq m \leq n-1 : d(\sigma^m(x), \sigma^m(y)) < t\}|.$$

Remark 1.1. In [1] the authors show that if there are $x, y \in \Sigma_P$, $t > 0$ such that $F_{xy}(t) < 1$ then $h_{\sigma_P} > 0$. If such x, y and t exist, then there is some $y' \in \Sigma_P$ such that $\bar{d}(y') > 0$. Because let $t = 2^{-l}$ then there exists an increasing sequence $\{q_i\}_{i=1}^\infty$ and $\epsilon > 0$ such that either $|\{0 \leq j \leq q_i : x_j \neq 0\}| > \frac{q_i \epsilon}{l+1}$ or $|\{0 \leq j \leq t_i : y_j \neq 0\}| > \frac{q_i \epsilon}{l+1}$. Hence $\bar{d}(x)$ or $\bar{d}(y)$ is positive.

In [1, Lemma 3.5], it has been proved that if $\mathbb{N} \setminus P$ contains an IP -set then $d(y) = 0$, for $y \in \Sigma_P$. We give a stronger result with a simpler proof.

Theorem 1.2. *If $\mathbb{N} \setminus P$ contains a Δ -set then $d^*(y) = 0$ for all $y \in \Sigma_P$.*

Proof. If $y \in \Sigma_P$, then $A_y - A_y \subset P$. But if there is y such that $d^*(y) > 0$ then $A_y - A_y$ is a Δ^* -set [5] and $\mathbb{N} \setminus P$ cannot have a Δ -set. \square

The following result is a reformulation of two results in [1].

Theorem 1.3. *If for all $y \in \Sigma_P$, $d(y) = 0$, then*

- (1) $h_{\sigma_P} = 0$,
- (2) σ_P is proximal.

Proof. (1) and (2) are proved in [1, Theorem 3.6] and [1, Theorem 3.11] respectively for the case when $\mathbb{N} \setminus P$ contains an IP -set. The proof of these theorems are based on the fact that if $\mathbb{N} \setminus P$ contains an IP -set then $d(y) = 0$, for any $y \in \Sigma_P$. Then this last result will lead to the both conclusions. \square

Again the proof of this Theorem is a minor alteration of in the proof of [1, Theorem 3.18].

Theorem 1.4. *There exists some $y \in \Sigma_P$ with $d^*(y) > 0$ if and only if $h_{\sigma_P} > 0$.*

Proof. First suppose there exists a point $y \in \Sigma_P$ such that $d^*(y) > 0$, so for some l there exist two increasing sequences $\{M_i\}_{i=1}^\infty$, $\{N_i\}_{i=1}^\infty$ and $\gamma > 0$ such that

$$|\{M_i \leq j \leq N_i : y_{[j, j+l]} \neq 0^{l+1}\}| \geq (N_i - M_i)\gamma.$$

So

$$|\{M_i \leq j \leq N_i : y_j \neq 0\}| \geq \frac{(N_i - M_i)\gamma}{l+1}.$$

Then by definition we have

$$h_{\sigma_P} \geq \lim_{N_i - M_i \rightarrow \infty} \frac{1}{N_i - M_i} \log(2^{\frac{(N_i - M_i)\gamma}{l+1}}) > 0.$$

Conversely, if for any $y \in \Sigma_P$, $d^*(y) = 0$ then $d(y) = 0$ and the proof follows from Theorem 1.3. \square

An immediate consequence of the above theorem is that if P is not Δ^* , then $h_{\sigma_P} = 0$. In particular, this sorts out the second question.

By Theorem 1.3, if $h_{\sigma_P} > 0$, then there is a $y \in \Sigma_P$ such that $d(y) > 0$. Combining this with the results of the above Theorem we have:

Corollary 1.5. *There is a point $y \in \Sigma_P$ with $d(y) > 0$ if and only if for some y' , $d^*(y') > 0$.*

The following gives an answer to the third question. Moreover, this result and the fact that when P misses an IP -set then it is not Δ^* and so has zero entropy is an answer for the first question as well.

Theorem 1.6. *If $h_{\sigma_P} = 0$ then Σ_P is proximal.*

Proof. Suppose $h_{\sigma_P} = 0$. Then by Theorem 1.4, for any $y \in \Sigma_P$ we have $d^*(y) = 0$ which implies that $d(\{i : y_i = 0\}) = 1$. Hence for any two points $x, y \in \Sigma_P$, $d(\{i : x_i = 0\} \cap \{i : y_i = 0\}) = 1$ and this in turn implies that Σ_P is proximal. \square

1.1. A necessary condition for transitivity. Still there is not a characterization for P to have Σ_P transitive. This also has been put as a question in [1, Question 1]. A necessity is the following.

Theorem 1.7. *Suppose Σ_P is transitive. Then P is an $IP - IP$ set.*

Proof. For any TDS such as (X, T) , the return times of a recurrence point x to any non-empty open set U , that is, $N(x, U) = \{n \in \mathbb{N} : T^n(x) \in U\}$ is an IP -set [6, Theorem 2.17]. Now let y be a transitive point. Then y is a recurrence point and $N(y, [1])$ is an IP -set. But $N(y, [1]) = \{y_i : y_i = 1\} = A_y$ and so $A_y - A_y \subset P$ and as a result P is an $IP - IP$ set. \square

An application of the above theorem is that any thick subset of natural numbers is an $IP - IP$ set. This is because Σ_P is weak mixing if and only if P is thick and if a TDS is weak mixing, then it is transitive, in fact, totally transitive: (Σ_P, σ^n) is transitive for all $n = 0, 1, \dots$

It is not hard to see that for any infinite subset of \mathbb{N} such as A , $P = FS(A) - FS(A)$ is a transitive system. On the other hand, let $k \geq 3$, $p_2 > p_1$ and $p_2 - p_1 \neq kn$ for any $n \in \mathbb{N}$. Now if $P = k\mathbb{N} \cup \{p_1, p_2\}$, then Σ_P is not transitive, however it is clearly $IP - IP$ set. Because it contains an $IP - IP$ set such as $k\mathbb{N}$.

By now we understand that this is the structure in P and not density which gives interesting dynamics to our spacing shifts systems. For instance, if P is not a Δ -set-set, then for all $y \in \Sigma_P$, $\sum_{i=1}^{\infty} y_i < \infty$. This gives a very simple dynamics to Σ_P . In fact, it is an equicontinuous system where any point will be attracted to 0^∞ eventually. We may choose P to have high density. As an example, for any $\epsilon > 0$ let $\frac{1}{k} < \epsilon$ and set $P = \mathbb{N} \setminus k\mathbb{N}$. Then $d(P) \geq 1 - \epsilon$ and since $k\mathbb{N}$ is a Δ^* -set P does not contain any Δ -set.

2. COMBINATORIAL CHARACTERIZATION FOR ZERO ENTROPY

In section 1, we showed that P must be at least Δ^* set, that is a highly structured and large set to have positive entropy. Here we show that even if P is a Δ^* set, it is not guaranteed that $h_{\sigma_P} > 0$.

One calls $E \subset \mathbb{N}$ a *density interseective* set if for any $A \subset \mathbb{N}$ with positive upper Banach density, $E \cap (A - A) \neq \emptyset$. For instance, any IP -set is a density interseective

set. In fact, if $R \subset \mathbb{N}$ is an IP -set and $p(\cdot)$ is a polynomial such that $p(\mathbb{N}) \subset \mathbb{N}$, then $E = \{p(n) : n \in R\}$ is a density intersective set [4].

Theorem 2.1. $h_{\sigma_P} = 0$ if and only if $P = \mathbb{N} \setminus E$ where E is a density intersective set.

Proof. Suppose $h_{\sigma_P} = 0$. If $E = \mathbb{N} \setminus P$ is not density intersective, then there must be a set A with positive upper Banach density such that $A - A \subseteq P$. Choose $y \in \Pi_{i=0}^{\infty}\{0, 1\}$ such that $y_i = 1$ if and only if $i \in A$. Then $y \in \Sigma_P$ and $A = A_y$. But this is absurd by Theorem 1.4.

For the other side, if E is density intersective, then P does not contain any $A - A$ where A is as above. Therefore, for all $y \in \Sigma_P$, $d^*(y) = 0$ which implies $h_{\sigma_P} = 0$. \square

It is an easy exercise to show that $\{n^2 : n \in \mathbb{N}\}$ does not contain any Δ -set. So $P = \mathbb{N} \setminus E$ is a Δ^* set and by the above theorem, $h_{\sigma_P} = 0$.

2.1. Positive entropy with no non-zero periodic points. Any spacing shift has 0^∞ as its periodic point. But a spacing shift has a non-zero periodic point of period k if and only if P contains $k\mathbb{N}$ [1, Lemma 2.6]. This implies there is a point y with $d(y) \geq \frac{1}{k}$ and so by Theorem 1.4 we have positive entropy.

Theorem 2.2. *There is P such that Σ_P has positive entropy with no non-zero periodic points.*

Proof. A theorem of Kříž [8] states that there is a set A with positive upper Banach density whose difference set contains no Bohr set. So let $y = \{y_i\}_{i \in \mathbb{N}}$ be defined by $y_i = 1$ if $i \in A$ and zero otherwise. Set $P = A - A$. Then $y \in \Sigma_P$, $A_y = A$ and $\bar{d}(y) = \bar{d}(A) > 0$. Therefore, $h_{\sigma_P} > 0$ and since P does not contain any Bohr set it does not contain any $k\mathbb{N}$ and the proof is complete. \square

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